Transient and Busy Period Analysis of the GI/G/1 Queue: Part II, Solution as a Hilbert Problem

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Abstract

In this paper we find the waiting time distribution in the transient domain and the busy period distribution of the $GI/G/1$ queue. We formulate the problem as a two dimensional Lindley process and then transform it to a Hilbert factorization problem. We achieve the solution of the factorization problem for the $GI/R/1, R/G/1$ queues, where R is the class of distributions with rational Laplace transforms. We obtain simple closed form expressions for the Laplace transforms of the waiting time distribution under FCFS when the system is initially empty and the busy period distribution. Furthermore, we find closed form formulae for the first two moments of the distributions involved.

Key words. Transient analysis, busy period, Lindley equation, Hilbert factorization.

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1 Introduction

In the first part of this work (Bertsimas and Nakazato [1]) we presented a method to perform transient and busy period analysis for the $MGE_L/MGE_M/1$ queue, where MGE is the class of mixed generalized Erlang distributions. Our analysis used the method of stages combined with the separation of variables and root finding techniques together with linear and tensor algebra. We found simple closed form expressions for the Laplace transforms of the queue length and the waiting time distribution under FCFS when the system is initially empty and the busy period distribution. In this paper we extend and generalize these results to the $GI/G/1$ queue with arbitrary distributions. We first formulate the problem as a two dimensional Lindley process and then transform it to a Hilbert factorization problem. We are able to solve explicitly the underlying factorization problem for the cases of $GI/R/1$ and $R/G/1$ queues, where R is the class of distributions with rational Laplace transforms. As a result, we find closed form formulae for the Laplace transforms of the waiting time and busy period distribution.

Formulations of queueing problems as Hilbert factorization problems can be traced back in Lindley [6], in which the steady state waiting time distribution of the $GI/G/1$ queue is derived via a spectral factorization of the underlying Hilbert problem. For other examples of the method see Keilson [2,3].

The paper is organized as follows. In the next section, which is central in the paper we formulate the transient behavior of the $GI/G/1$ queue as a two dimensional Lindley process, derive the key formula of the transient and busy period dynamics and then transform it to a Hilbert factorization problem. In Section 3, we solve the factorization problem for the $R/G/1$ queue, while in Section 4 we achieve its solution for the $GI/R/1$ queue. In Section 5 we observe how the results of the previous two sections are in agreement with the known results for the $M/G/1$ and $GI/M/1$ queues and consistent with the results of Bertsimas and Nakazato [1]. The final section contains some closing remarks.

2 System Formulation

In this section we formulate the transient behavior of the $GI/G/1$ queue as a two dimensional Lindley process, derive the key formula of the transient dynamics and then transform it to a Hilbert factorization problem. Our analysis will focus on the notion of a *busy interval*, which is defined as the busy period plus an immediately following idle period. In Subsection 2.1 we define the notation we will use, in Subsection 2.2 we derive the key formula for the transient dynamics and in Subsection 2.3 we transform the problem to a Hilbert factorization problem.

2.1 Notation and Assumptions

In this subsection we define the random variables and establish the notation we are using. We assume that the system is initially idle and the first customer's arriving time is the forward recurrence interarrival time. Although this assumption is restrictive for the waiting time distribution, it is not restrictive for the busy period distribution, since the busy period regenerates.

We first define the random variables we will use as follows:

X_n : the service time of n th customer.

T_n : the interarrival time between $n-1$ th and n th customer.

τ_n : the arriving time of n th customer. Note that $\tau_n = \tau_1 + \sum_{k=2}^{n} T_k$.

τ : the arriving time of a random customer.

B_I : the duration of a busy interval, i.e. the interval between the initiating epoch of a busy period and the initiating epoch of the next busy period.

B_P : the duration of a busy period.

W_n^+ : the waiting time in the queue of n th customer.

W^+ : the waiting time of a random customer.

We will use the following notation:

$a(t)$: the interarrival time probability density function (pdf).

$\alpha(s)$: the Laplace transform of $a(t)$.

$\frac{1}{\lambda} = \mathrm{E}[T_n] = -\dot{\alpha}(0)$: the mean interarrival time.

$C_T^2 = \mathrm{Var}[T_n]/\,\mathrm{E}[T_n]^2$: the squared coefficient of variation of the interarrival time.

$a^*(t)$: the first customer's arriving time pdf (because of our assumption it is the forward recurrence time of the interarrival time).

$\alpha^*(s)$: the Laplace transform of $a^*(t)$, i.e. $\alpha^*(s) = \frac{\lambda}{s}(1 - \alpha(s))$.

$b(t)$: the service time pdf.

$\beta(s)$: the Laplace transform of $b(t)$.

$\frac{1}{\mu} = \mathrm{E}[X_n] = -\dot{\beta}(0)$: the mean service time.

$C_X^2 = \mathrm{Var}[X_n]/\,\mathrm{E}[X_n]^2$: the squared coefficient of variation of the service time.

$\rho = \frac{\lambda}{\mu}$: the traffic intensity.

$s_I(t)$: the busy interval pdf.

$s_P(t)$: the busy period pdf.

$\sigma(s)$: the Laplace transform of $s_P(t)$.

In addition, we define

$$\begin{aligned} f(x,y) &= \frac{\partial}{\partial y}\Pr[W^+ \le y | \tau = x] \\ &= \lim_{N \to \infty} \frac{\frac{1}{N}\sum_{n=1}^{N}\frac{\partial^2}{\partial x \partial y}\Pr[\tau_n \le x, W_n^+ \le y]}{\frac{1}{N}\sum_{n=1}^{N}\frac{d}{dx}\Pr[\tau_n \le x]} \\ &= \frac{\frac{\partial^2}{\partial x \partial y}\sum_{n=1}^{\infty}\Pr[\tau_n \le x, W_n^+ \le y]}{\frac{d}{dx}\sum_{n=1}^{\infty}\Pr[\tau_n \le x]}. \qquad (1) \end{aligned}$$

2.2 Transient Dynamics

In this subsection we derive the key formula that describes the transient dynamics of the $GI/G/1$ queue. For notational convinience we enumerate customers by $0, 1, 2, \ldots, n$ in the order of arrival. We analyze the case, in which the n th customer arrives at the busy period initiated by k th customer. We let

$$\begin{aligned} \xi_n &= X_{n-1} - T_n \\ W_n &= \sum_{r=k+1}^{n} \xi_r \end{aligned}$$

and observe (see Figure 1) that if $W_{n+k} \leq 0$ and $W_r > 0$ for $r = k+1 \ldots n+k-1$ then

$$B_I = \sum_{r=k+1}^{k+n} T_r = \tau_{n+k} - \tau_k$$

$$B_P = \sum_{r=k}^{k+n-1} X_r = \sum_{r=k+1}^{k+n} (T_r + \xi_r) = B_I + W_{n+k}. \tag{2}$$

Similarly,

$$\text{if} \quad W_r > 0 \quad \text{for} \quad r = k+1 \ldots n+k, \quad \text{then} \quad W_{n+k}^{+} = W_{n+k}. \tag{3}$$

· Summarizing, the critical observation is that if $W_{n+k} \leq 0$, then the idle period, that immediately follows the busy period B_P, is $-W_{n+k}$; on the other hand, if $W_{n+k} > 0$, then W_{n+k} is waiting time of $k+n$ th customer. Therefore, if we keep track of the busy interval B_I and the quantity W_{n+k}, then we can find both the busy period and the waiting time from (2) and (3) respectively. For this goal we now consider the joint densities:

$$\begin{aligned} \Delta(x, y) &= \frac{\partial^2}{\partial x \partial y} Pr\{T_n \leq x, \xi_n \leq y\}, \\ f_n(x, y) &= \frac{\partial^2}{\partial x \partial y} Pr\{\tau_{n+k} - \tau_k \leq x, W_{n+k}^{+} \leq y, W_r > 0, r = k+1 \ldots n+k\}, \\ f_0(x, y) &= \delta(x)\delta(y), \end{aligned}$$

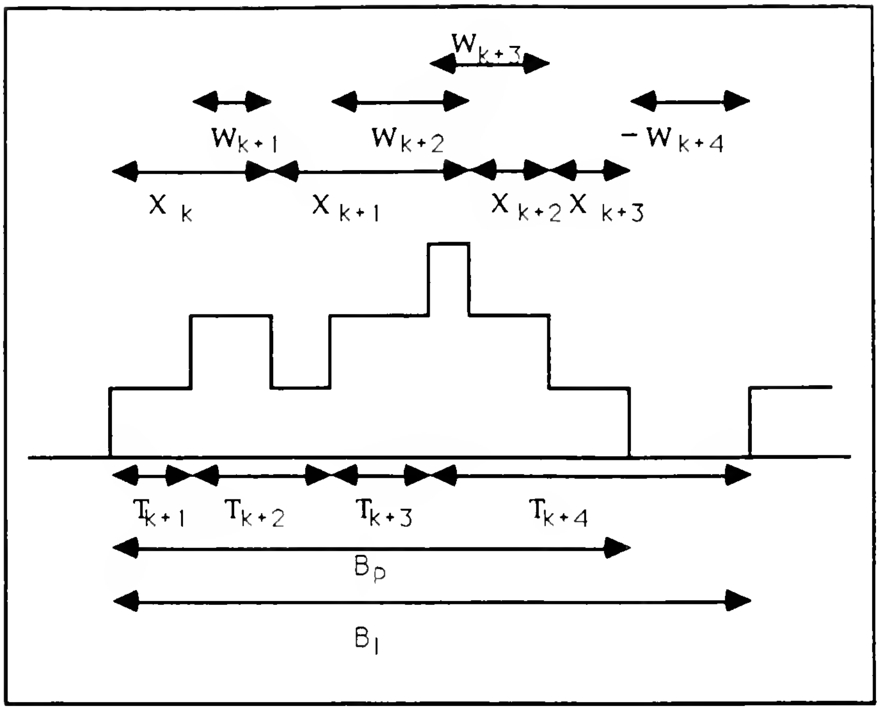

Figure 1: Transient dynamics

where $\delta(x)$ is the Dirac delta function. Note that that $\Delta(x, y)$ is independent of n and $f_n(x, y)$ has positive support in y, nonnegative support in x and is independent of k.

Since $\tau_{n+k+1} - \tau_k = \tau_{n+k} - \tau_k + T_{n+k+1}$ and $W^+_{n+k+1} = W^+_{n+k} + \xi_{n+k+1}$ if $W_r > 0, r = k + 1 \ldots n + k + 1$ we obtain the recurrence relations:

$$
\begin{aligned}
f_0(x, y) &= \delta(x)\delta(y) \\
f_1(x, y) &= \Delta(x, y)U(y) \\
&\vdots \\
f_{n+1}(x, y) &= [f_n(x, y) * \Delta(x, y)]U(y), \qquad (4)
\end{aligned}
$$

where $U(y)$ is a unit step function and we denote "*" as the 2-dimensional convolution sign that is $f_n(x, y) * \Delta(x, y) \triangleq \int_{-\infty}^{\infty} \int_0^x f_n(x - u, y - v)\Delta(u, v)\, du\, dv$. We also define

$$r_n(x,y) = \frac{\partial^2}{\partial x \partial y} Pr\{B_I \le x, W_{n+k} \le y, W_r > 0, r = k+1 \dots n+k-1, W_{n+k} \le 0\}.$$

Note that $r_n(x,y)$ has nonpositive support in y and nonnegative support in x and it is independent of k.

The motivation for the above definitions is that we can express the pdf of the quantities of interest in terms of the functions $r_n(x,y)$. Clearly

$$s_I(x) \triangleq \frac{d}{dx} Pr\{B_I \le x\} = \int_{-\infty}^{0} \sum_{n=1}^{\infty} r_n(x,y)\, dy, \tag{5}$$

and using (2)

$$s_P(x) \triangleq \frac{d}{dx} Pr\{B_P \le x\} = \int_{-\infty}^{0} \sum_{n=1}^{\infty} r_n(x-y,y)\, dy. \tag{6}$$

Using (2) and (3) we obtain in a similar way as before

$$\begin{aligned} r_1(x,y) &= \Delta(x,y)(1-U(y)) \\ &\vdots \\ r_{n+1}(x,y) &= [f_n(x,y) * \Delta(x,y)](1-U(y)). \end{aligned} \tag{7}$$

From (4) and (7) we obtain the key formula for the $GI/G/1$ transient dynamics in real time:

$$f_{n+1}(x,y) + r_{n+1}(x,y) = f_n(x,y) * \Delta(x,y). \tag{8}$$

2.3 Formulation as a Hilbert Problem

In this subsection we will work in the transform domain, where the solution of (8) is equivalent to a Hilbert factorization problem. We introduce the Laplace transforms:

$$\Phi^+(s,\omega) = \int_0^\infty \int_0^\infty e^{-sx-\omega y} \sum_{n=0}^{\infty} f_n(x,y)\, dx\, dy,$$

$$\rho^-(s,\omega) = \int_{-\infty}^{0} \int_0^\infty e^{-sx-\omega y} \sum_{n=1}^{\infty} r_n(x,y)\, dx\, dy.$$

Note that

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-sx-\omega y}\Delta(x,y)\,dx\,dy = \alpha(s-\omega)\beta(\omega).$$

The superscript + is employed to designate that $\Phi^{+}(s,\omega)$ is analytic in the right half of the complex ω plane. Similarly, the superscript − designates that $\rho^{-}(s,\omega)$ is analytic in the left half of the complex ω plane.

By taking transforms in (8) we obtain

$$\Phi^{+}(s,\omega) + \rho^{-}(s,\omega) = 1 + \alpha(s-\omega)\beta(\omega)\Phi^{+}(s,\omega),$$

or equivalently

$$\Phi^{+}(s,\omega)(1-\alpha(s-\omega)\beta(\omega)) = 1-\rho^{-}(s,\omega). \tag{9}$$

(9) is a Hilbert factorization problem in ω with fixed s, where

$\Phi^{+}(s,\omega)$ is analytic in $\mathrm{Re}(\omega) > 0$ and $\mathrm{Re}(s) > 0$

$\rho^{-}(s,\omega)$ is analytic in $\mathrm{Re}(\omega) < 0$ and $\mathrm{Re}(s) > 0$.

The following additional boundary conditions complete the description of the factorization problem:

$$\Phi^{+}(s,\infty) = 1 \quad \left(\Leftrightarrow \int_{0^-}^{0^+}\sum_{n=0}^{\infty} f_n(x,y)\,dy = \delta(x)\right)$$

$$\dot{\alpha}(0) < \dot{\beta}(0) \qquad\qquad (\Leftrightarrow \rho < 1)$$

Once $\rho^{-}(s,\omega)$ is found, we can use (6) to obtain the Laplace transform of the busy period:

$$\sigma(s) \triangleq \int_{0}^{\infty} e^{-sx} s_P(x)\,dx = \rho^{-}(s,s), \tag{10}$$

and similarly from (5)

$$\int_{0}^{\infty} e^{-sx} s_I(x)\,dx = \rho^{-}(s,0).$$

The transform of the conditional waiting time (transform variable ω) in the queue of a customer whose arriving time (transform variable s) is given, can be found from $\Phi^+(s,\omega)$ as follows. From (1) we find that (the convolution "*" is with respect to x)

$$\begin{aligned} f(x,y) &= \frac{\partial}{\partial y}\Pr[W^+ \le y | \tau = x] \\ &= \frac{1}{\lambda} a^*(x) * \sum_{r=0}^{\infty} s_I^{(r)}(x) * \sum_{n=0}^{\infty} f_n(x,y), \end{aligned} \qquad (11)$$

since we assumed that the arriving time of the first customer is the forward recurrence interarrival time and thus from the renewal theorem (or my simply taking Laplace transforms) we have

$$\frac{d}{dx}\sum_{n=1}^{\infty}\Pr[\tau_n \le x] = a^*(x) * \sum_{n=0}^{\infty} a^{(n)}(x) = \lambda,$$

and moreover

$$\frac{\partial^2}{\partial x \partial y}\sum_{n=1}^{\infty}\Pr[\tau_n \le x, W_n^+ \le y] = a^*(x) * \sum_{r=0}^{\infty} s_I^{(r)}(x) * \sum_{n=0}^{\infty} f_n(x,y).$$

By defining

$$\Phi(s,\omega) = \int_0^{\infty}\int_0^{\infty} e^{-sx-\omega y} f(x,y)\,dx\,dy$$

and taking transforms in (11) we obtain that

$$\begin{aligned} \Phi(s,\omega) &= \frac{\alpha^*(s)}{\lambda(1-\rho^-(s,0))}\Phi^+(s,\omega) \\ &= \frac{\Phi^+(s,\omega)}{s\Phi^+(s,0)}. \end{aligned} \qquad (12)$$

Therefore, we can express both the transforms of the busy period and the waiting time distribution in terms of $\Phi^+(s,\omega)$ and $\rho^-(s,\omega)$. As a result, we reduced the problem of obtaining the transforms of the busy period and the waiting time distribution to the solution of the Hilbert problem (9).

In its full generality, i.e., with completely arbitrary interarrival and service time distributions, it is not known whether the Hilbert problem (9) has a closed form solution. In special cases, however, when one of the distributions has a rational Laplace transform, then we can solve the factorization problem in closed form. In the next sections we solve (9) for the $R/G/1$ and $GI/R/1$ respectively, where R is the class of distributions with rational Laplace transforms.

3 The Solution of the Hilbert Problem for the $R/G/1$ Queue

In this case $\alpha(s) = \frac{\alpha_N(s)}{\alpha_D(s)}$, where $\alpha_D(s)$ is a monic polynomial in s of degree L and $\alpha_N(s)$ is a polynomial of degree less than L.

For fixed s with $\text{Re}(s) \geq 0$, let $z = x_r(s)$, $(r = 1 \ldots L)$ be the L roots of the equation:

$$\alpha(s-z)\beta(z) = 1, \qquad \text{Re}(z) \geq 0. \tag{13}$$

The proof of this follows along the lines of claim 3 of [1]. Once the number of roots is established through Rouche's theorem, we simply follow the methods pioneered by Keilson [3,2]. Now, (9) can be written as

$$\frac{\Phi^+(s,\omega)}{\frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega)-\alpha_N(s-\omega)\beta(\omega)}} = \frac{1-\rho^-(s,\omega)}{\frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega)}}. \tag{14}$$

By observing that the expression in the rhs of the equation (14) is analytic for $\text{Re}(\omega) > 0$ and the expression in the lhs of the the equation (14) is analytic for $\text{Re}(\omega) < 0$ and using Liouville's theorem we conclude that both expressions should be equal to a function of s. From the boundary conditions of (9) we easily find that the function is a constant function 1. To complete Liouville's theorem, we need the following proposition.

Proposition 1 *The expressions in both sides of the equation (14) are bounded.*

Proof

Let $\text{Re}(s) \geq 0$. For the lhs, with $\text{Re}(\omega) \geq 0$, it is easily seen (since the zeros cancel out) that the denominator is bounded away from 0, and thus for some $\epsilon > 0$;

$$\left|\frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega)-\alpha_N(s-\omega)\beta(\omega)}\right| \geq \epsilon.$$

We then check that the numerator is also bounded;

$$|\Phi^+(s,\omega)| \leq \Phi^+(0,0)$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \sum_{n=0}^{\infty} f_n(x,y)\,dx\,dy \\
&\le \sum_{n=0}^{\infty} \left| \int_0^\infty \int_0^\infty f_n(x,y)\,dx\,dy \right| \\
&\le 1 + \sum_{n=1}^{\infty} Pr\{ \sum_{r=k+1}^{k+n} \xi_r > 0 \}
\end{aligned}
$$

Since $\rho < 1$ $E[\xi_r] < 0$ $(\forall r)$. As a result, applying the Chernoff bound, we obtain that there exists a constant $\delta < 1$ such that $Pr\{\sum_{r=k+1}^{k+n} \xi_r > 0\} < \delta^n$, and thus

$$
|\Phi^+(s,\omega)| \le \frac{1}{1-\delta} < \infty.
$$

In an analogous way the denominator of the rhs, with $\mathrm{Re}(\omega) \le 0$, is bounded away from 0, i.e., for some $\epsilon > 0$;

$$
\left| \frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega)} \right| \ge \epsilon.
$$

In addition the boundness of the numerator of the rhs is seen as follows;

$$
\begin{aligned}
|1-\rho^-(s,\omega)| &\le 1 + |\rho^-(s,\omega)| \\
&\le 1 + \rho^-(0,0) = 1 + \sigma(0) = 2 \\
&< \infty. \qquad \square
\end{aligned}
$$

Thus by applying Liouville 's theorem we conclude that the unique solution to the Hilbert factorization problem (9) is:

$$
\begin{aligned}
\Phi^+(s,\omega) &= \frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega) - \alpha_N(s-\omega)\beta(\omega)} \\
\rho^-(s,\omega) &= 1 - \frac{\prod_{r=1}^{L}(x_r(s)-\omega)}{\alpha_D(s-\omega)}.
\end{aligned}
$$

Hence we get from (12)

$$
\Phi(s,\omega) = \frac{\alpha_D(s) - \alpha_N(s)}{s\,(\alpha_D(s-\omega) - \alpha_N(s-\omega)\beta(\omega))} \prod_{r=1}^{L} \frac{x_r(s)-\omega}{x_r(s)} \tag{15}
$$

and from (10)

$$
\sigma(s) = 1 - \frac{1}{\alpha_D(0)} \prod_{r=1}^{L} (x_r(s) - s). \tag{16}
$$

The reward of our analysis is a simple closed form expression for the transform of the busy period and waiting time distribution. Moreover, we can find closed form expressions for the first two moments of the waiting time and busy period distribution by differentiating the corresponding transforms. The following formulae were derived using the symbolic differentiation routine of the software package Mathematica on a Macintosh II computer.

$$\begin{aligned}\int_0^\infty e^{-sx}\,\mathrm{E}[W^+|\tau = x]\,dx &= -\lim_{\omega\to 0}\frac{\partial}{\partial\omega}\Phi(s,\omega)\\ &= \frac{1}{s}\left(\sum_{r=1}^{L}\frac{1}{x_r(s)} + \frac{\frac{1}{\mu}\alpha(s)+\dot\alpha(s)}{1-\alpha(s)} - \frac{\dot\alpha_D(s)}{\alpha_D(s)}\right)\end{aligned}$$

$$\begin{aligned}&\int_0^\infty e^{-sx}\,\mathrm{E}[(W^+)^2|\tau = x]\,dx = \lim_{\omega\to 0}\frac{\partial^2}{\partial\omega^2}\Phi(s,\omega)\\ &= \tfrac{1}{s}\Bigg(-\textstyle\sum_{r=1}^{L}\frac{1}{x_r(s)^2} + \left(\frac{\dot\alpha_D(s)}{\alpha_D(s)}\right)^2 - \frac{\ddot\alpha_D(s)}{\alpha_D(s)}\\ &\quad + \frac{(1+C_X^2)\alpha(s) - C_X^2\alpha(s)^2 + 2\mu\dot\alpha(s) + \mu^2\dot\alpha(s)^2 + \mu^2\ddot\alpha(s) - \mu^2\alpha(s)\ddot\alpha(s)}{\mu^2(1-\alpha(s))^2}\\ &\quad + \left(\textstyle\sum_{r=1}^{L}\frac{1}{x_r(s)} + \frac{\frac{1}{\mu}\alpha(s)+\dot\alpha(s)}{1-\alpha(s)} - \frac{\dot\alpha_D(s)}{\alpha_D(s)}\right)^2\Bigg)\end{aligned}$$

$$\begin{aligned}\mathrm{E}[B_P] &= -\lim_{s\to 0}\frac{d}{ds}\sigma(s) = \frac{\rho}{(1-\rho)\alpha_D(0)}\prod_{r=1}^{L-1}x_r(0)\\ \mathrm{Var}[B_P] &= \lim_{s\to 0}\frac{d^2}{ds^2}\log(\sigma(s))\\ &= \frac{\rho}{(1-\rho)\alpha_D(0)}\sum_{r=1}^{L-1}\frac{\alpha(-x_r(0))\dot\beta(x_r(0))}{\dot\alpha(-x_r(0))\beta(x_r(0)) - \alpha(-x_r(0))\dot\beta(x_r(0))}\prod_{\substack{k=1\\k\neq r}}^{L-1}x_k(0)\\ &\quad - \frac{\rho^2(C_T^2+C_X^2)}{(1-\rho)^3\lambda\alpha_D(0)}\prod_{r=1}^{L-1}x_r(0) - \left(\frac{\rho}{(1-\rho)\alpha_D(0)}\right)^2\prod_{r=1}^{L-1}x_r(0)^2,\end{aligned}$$

where we used $\frac{d}{ds}(s - x_r(s)) = \frac{\alpha(s-x_r(s))\dot\beta(x_r(s))}{\alpha(s-x_r(s))\dot\beta(x_r(s)) - \dot\alpha(s-x_r(s))\beta(x_r(s))}$,
and $\frac{d^2}{ds^2}(s - x_r(s)) = \alpha(s - x_r(s))\beta(x_r(s)) \times$
$\frac{\alpha(s - x_r(s))\ddot\alpha(s - x_r(s))\dot\beta(x_r(s))^2 - 2\dot\alpha(s - x_r(s))^2\dot\beta(x_r(s))^2 + \dot\alpha(s - x_r(s))^2\beta(x_r(s))\ddot\beta(x_r(s))}{\left(\alpha(s - x_r(s))\dot\beta(x_r(s)) - \dot\alpha(s - x_r(s))\beta(x_r(s))\right)^3}$.

The formula for the first two moments of the busy period was simplified using the observation that there exists a unique root such that $x_L(0) = 0$ (see Keilson [4]).

As an additional check of the algebra we can verify that for the $M/G/1$ queue, i.e. $L = 1$ the formula for $E[B_P]$ becomes $E[B_P] = \frac{1}{\mu-\lambda}$. Finally, note that the roots $x_r(0)$ are precisely the roots that appear in the steady solution of the $R/G/1$ queue.

4 The Solution of the Hilbert Problem for the $GI/R/1$ Queue

In this case $\beta(s) = \frac{\beta_N(s)}{\beta_D(s)}$, where $\beta_D(s)$ is a monic polynomial in s of degree M and $\beta_N(s)$ is a polynomial of degree less than M.

As in the previous section, for fixed s with $\text{Re}(s) \geq 0$, let $z = x_r(s)$ $(r = 1 \ldots M)$ be the M roots of the equation:

$$\alpha(s-z)\beta(z) = 1, \qquad \text{Re}(z) < 0.$$

The unique solution to the Hilbert problem can be found in a similar way as in the previous section to be:

$$\begin{aligned} \Phi^+(s,\omega) &= \frac{\beta_D(\omega)}{\prod_{r=1}^{M}(\omega - x_r(s))} \\ \rho^-(s,\omega) &= 1 - \frac{\beta_D(\omega) - \alpha(s-\omega)\beta_N(\omega)}{\prod_{r=1}^{M}(\omega - x_r(s))}. \end{aligned}$$

Note that the connection with the results of the previous section in the case of $R/R/1$ is established by noticing that

$$(-1)^L \prod_{r=1}^{L+M} (\omega - x_r(s)) = \alpha_D(s-\omega)\beta_D(\omega) - \alpha_N(s-\omega)\beta_N(\omega).$$

Hence we get from (10) and (12) that

$$\Phi(s,\omega) = \frac{\beta_D(\omega)}{s\beta_D(0)} \prod_{r=1}^{M} \frac{x_r(s)}{x_r(s) - \omega} \tag{17}$$

$$\sigma(s) = 1 - \frac{\beta_D(s) - \beta_N(s)}{\prod_{r=1}^{M}(s - x_r(s))}. \tag{18}$$

As an accuracy check we can easily check that (17) and (18) are identical with the results for the $MGE_L/MGE_M/1$ queue obtained in part I of this study (Bertsimas and Nakazato [1]). As in the previous section we can find closed form formulae for the moments of the distributions involved as follows:

$$\int_0^\infty e^{-sx}\,\mathrm{E}[W^+|\tau = x]\,dx = -\frac{1}{s}\left(\sum_{r=1}^{M}\frac{1}{x_r(s)} + \frac{\dot{\beta}_D(0)}{\beta_D(0)}\right)$$

$$\int_0^\infty e^{-sx}\,\mathrm{E}[(W^+)^2|\tau = x]\,dx = \frac{1}{s}\left(\sum_{r=1}^{M}\frac{1}{x_r(s)^2} + \frac{\ddot{\beta}_D(0)}{\beta_D(0)} - \left(\frac{\dot{\beta}_D(0)}{\beta_D(0)}\right)^2 + \left(\sum_{r=1}^{M}\frac{1}{x_r(s)} + \frac{\dot{\beta}_D(0)}{\beta_D(0)}\right)^2\right)$$

$$\mathrm{E}[B_P] = \frac{(-1)^M\beta_D(0)}{\mu}\prod_{r=1}^{M}\frac{1}{x_r(0)}$$

$$\begin{aligned}\mathrm{Var}[B_P] = &\left(\frac{(-1)^M\beta_D(0)}{\mu}\sum_{k=1}^{M}\frac{\alpha(-x_k(0))\dot{\beta}(x_k(0))}{x_k(0)\{\dot{\alpha}(-x_k(0))\beta(x_k(0)) - \alpha(-x_k(0))\dot{\beta}(x_k(0))\}}\right.\\ &\left.+\frac{(-1)^M(1+C_X^2)\beta_D(0)}{\mu^2} - \frac{\dot{\beta}_D(0)}{\mu}\right)\prod_{r=1}^{M}\frac{1}{x_r(0)} - \left(\frac{\beta_D(0)}{\mu}\right)^2\prod_{r=1}^{M}\frac{1}{x_r(0)^2}.\end{aligned}$$

5 The $M/G/1$ and $GI/M/1$ Queues

In this section we verify and generalize well known results for the $GI/M/1$ and $M/G/1$ queues.

For the $GI/M/1$ it is known (Takacs [7]) that $\sigma(s) = \frac{\mu(1-w(s))}{s+\mu-w(s)}$, where $w(s) = a(s - x_1(s))$. By letting $M = 1$ in (18) and observing that $w(s)\beta(x_1(s)) = 1$, i.e. $x_1(s) = \mu(w(s) - 1)$, we find the same expression.

For the $M/G/1$ queue it is well known (see Kleinrock [5]) that the busy period satisfies $\sigma(s) = \beta(s + \lambda - \lambda\sigma(s))$. In order to see how we can derive this from (16) we observe that from (16) $\sigma(s) = 1 + \frac{s-x_1(s)}{\lambda}$, from where $x_1(s) = s + \lambda - \lambda\sigma(s)$. Since $x_1(s)$ satisfies from (13)

$$\alpha(s - x_1(s))\beta(x_1(s)) = \frac{\lambda}{\lambda + s - x_1(s)}\beta(s + \lambda - \lambda\sigma(s)) = 1,$$

we can now easily derive the desired relation $\sigma(s) = \beta(s + \lambda - \lambda\sigma(s))$.

The time-dependent behavior of the waiting time can be expressed in terms of $\sigma(s)$ as follows:

$$\Phi(s,\omega) = \frac{1}{s + \lambda(1 - \sigma(s))}\frac{s - \omega + \lambda(1 - \sigma(s))}{s - \omega + \lambda(1 - \beta(\omega))}$$

This is a solution to the well known Takacs integrodifferential equation (see Kleinrock [5] or Takacs [8]).

6 Concluding Remarks

In this paper we attempted to demonstrate the power of direct probabilistic arguments for the waiting time distribution in the transient domain and the busy period distribution for the $GI/G/1$ queue. We found closed form expressions for the transforms and the first two moments of these distributions. Algorithmically our approach offers a method for finding these distributions in the time domain through the numerical inversion of the Laplace transforms. In Bertsimas and Nakazato [1] we reported numerical results for finding numerically the busy period, the transient queue length and the waiting time distributions in a $MGE/MGE/1$ queue, by inverting numerically the corresponding Laplace transforms.

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